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Supercapacitor Modelling by Lagrange's Equations

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Abstract. A model of a double layer capacitor based on Lagrange's equations is proposed. The Lagrange function for the equivalent electric circuits is obtained by generalized charge coordinates. The model is implemented in Simulink® environment. Simulation results are compared with experimental test performed on a commercial supercapacitor of 83F@48V rated value.

Key words

Supercapacitor, Lagrange's equations, generalised coordinates, modelling, reliability.

1. Introduction

Nowadays, the use of supercapacitors (SCs) as fast and efficient energy storage solution in power application is widely recognized since they offer power densities typically larger than the ones achievable with traditional batteries and energy densities 10 to 20 times larger than electrolytic capacitors. The so-called Double-Layer-Capacitors (DLCs) are available on the market with capacitance values up to 1500 F and rated voltage of about 2.3 V; higher rated voltages are obtained by suitable series-parallel combinations of such single DLC units [1].
DLCs use polymer foils, which are able to offer a great density of electric charges increasing the capacitance value, on the other hand, this implies a non-linear behaviour of the relationship charge-voltage, differently form a classical capacitor. In the literature, different models based on electrochemical laws [2]-[3], equivalent electrical circuits [4]-[5], and model based on impedance measurement [6]-[7], are proposed. Among these models, the simplified physical model, proposed in [4], is suitable for practical engineering applications, over the range of some tens of minutes [1].
The above cited applications justify further studies on the DLC modelling and control to define models that are simple to be implemented and require a low computation time as that proposed in this work. It is based on Lagrange's equations, an example of implementation is performed in Simulink® environment. This approach uses the energy formulation to obtain the differential equations using the charge as generalized coordinate. It is not commonly adopted for linear circuits analysis where the Kirchhoff equations applied to meshes and nodes is by far the preferred tool. Nevertheless there are some advantages related to the use of the Lagrangian approach: it has a general validity, independent of the specific field of application, mechanical or electrical; it can be easily applied when non-linear components are present; it is very beneficial when modelling systems with interaction between electrical and mechanical parts. Finally it is a good starting point for state-variable equation.
The proposed implementation can be performed without the use of specific circuit simulator obtaining a reduced computation time.

2. Model of the Double Layer Capacitor

The 2-branches model is commonly used for power applications, it is composed of a branch with a voltage dependent capacitor series connected with a resistor and a branch with a linear resistor series connected with a linear capacitor. [1]-[6]. The leakage resistance models the self-discharge behaviour. Figure 1 shows a scheme of the DLC equivalent circuit. It can be noticed that the first branch contains the voltage dependent capacitor which is modelled by a linear capacitor which is connected in parallel to a linear voltage dependent capacitor. It causes the nonlinear behaviour of the DLC. The two voltages V_1 and V_2 are not directly measurable since they do not coincide with the voltage V at the external terminals due to the presence of the series parasitic resistance in each branch.

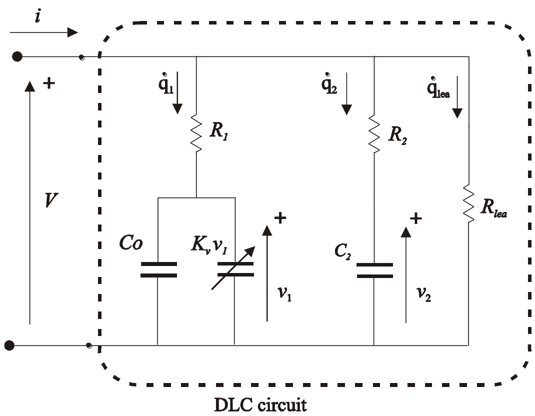

Fig 1: equivalent circuit of the supercapacitor

In the proposed approach, the charges $\{ q_1, q_2, q_{lea} \}$ are the generalized coordinates and the currents, as the time derivative of charge, represent the generalized velocities. It can be noted that q_1 and q_2 are state variables as well

whereas q_{lea} is the integral of the current flowing through the leakage resistance.
The capacitor C_1 of the first branch is linear dependent on the voltage following the law:

$$C_1 = C_0 + k_c V_1 \tag{1}$$

The stored charge is given by

$$q_1 = (C_0 + k_c V_1)V_1 = C_0 V_1 + k_c V_1^2 \tag{2}$$

The capacitor of the second branch C_2 is a linear capacitor described by the eq.

$$q_2 = C_2 V_2 \tag{3}$$

3. Lagrangian function for electric circuits

The Lagrangian function in generalized coordinates of charge is given by:

$$L_e = C_e - P_e \tag{4}$$

Where C_e is the kinetic electric co-energy and P_e is the electric potential energy. The lagrangian function, in terms of the generalized coordinates of charge and their derivatives, is given by:

$$L_e(q_k, \dot{q}_k) = W_i^*(\dot{q}_k) - W_c(q_k) \tag{5}$$

Where W_i^* is the co-energy stored in inductive elements and W_c is the energys stored in the capacitive elements. The Lagrange's equations for the electric circuits are obtained from:

$$\frac{d}{dt}\left(\frac{\partial L_e}{\partial \dot{q}_k}\right) - \left(\frac{\partial L_e}{\partial q_k}\right) + \left(\frac{\partial D_e}{\partial \dot{q}_k}\right) = V_k \tag{6}$$

Where D_e is the dissipation function. The V_k terms correspond to the generalized electric forces or generalized voltages. In the studied case only linear resistances dissipate energy, hence the dissipation function is:

$$D_e = \frac{1}{2}\sum_k R_k \dot{q}_k^2 \tag{7}$$

Where k={1,2,lea}.

4. Lagrange's equations for DLC

The Lagrange's equations for the supercapacitor model described in figure 1 can be derived considering that energy is stored only in capacitive elements, hence it is sufficient to evaluate the energy terms related to C_1 and C_2 capacitors and the dissipative function.
As far as the capacitor C_1 is concerned, from (2) the following equation is obtained:

$$V_1^2 k_c + V_1 C_0 - q_1 = 0 \tag{7}$$

Solving (7) and considering only the positive solution, as occurs in a real supercapacitor, the voltage at the terminals of C_1 is calculated.

$$V_1(q_1) = \frac{-C_0 \pm \sqrt{C_0^2 + 4k_c q_1}}{2k_c} \tag{8}$$

$$V_1(q_1) = \frac{C_0}{2k_c}\left(-1 + \sqrt{1 + \frac{4k_c}{C_0{}^2} q_1}\right) \tag{9}$$

Integrating (9) respect to the charge the energy is then obtained.

$$W_{C1}(q_1) = \int_0^q V_1(q)dq = \frac{C_0}{2k_c}\left[-q_1 + \int_0^q \left(1 + \frac{4k_c}{C_0^2} q\right)^{1/2} dq\right] =$$

$$= \frac{C_0}{2k_c}\left[-q_1 + \frac{C_0^2}{6k_c}\left(1 + \frac{4k_c}{C_0^2} q_1\right)^{3/2}\right] \tag{10}$$

As for the linear capacitor C_2, the energy is given by:

$$W_{C2}(q_2) = \int_0^q V(q)dq = \frac{1}{C_2}\int_0^q q dq = \frac{1}{2}\frac{q_2^2}{C_2} \tag{11}$$

Finally the Lagrangian function is obtained as:

$$L_e(q_1, q_2) = -[W_{C1}(q_1) + W_{C2}(q_2)] \tag{12}$$

Taking into account that the Lagrangian function does not depend on the derivative of the charge.

$$\frac{d}{dt}\left(\frac{\partial L_e}{\partial \dot{q}_k}\right) = 0 \tag{13}$$

The following terms are then obtained by (10),(11) and (12):

$$\left(\frac{\partial L_e}{\partial q_1}\right) = -\frac{-C_0 \pm \sqrt{C_0^2 + 4k_c q_1}}{2k_c} \tag{14}$$

$$\left(\frac{\partial L_e}{\partial q_2}\right) = -\frac{q_2}{C_2} \tag{15}$$

$$\left(\frac{\partial D_e}{\partial \dot{q}_1}\right) = R_1 \dot{q}_1 \tag{16}$$

$$\left(\frac{\partial D_e}{\partial \dot{q}_2}\right) = R_1 \dot{q}_2 \tag{17}$$

$$\left(\frac{\partial D_e}{\partial \dot{q}_l}\right) = R_{lea} \dot{q}_{lea} \tag{18}$$

Substituting eqs. (13)-(18) into eq. (6) for k={1,2,lea}, the Lagrange's equations are obtained.

$$\begin{cases} \dfrac{-C_0 \pm \sqrt{C_0^2+4k_c q_1}}{2k_c} + R_1\dot{q}_1 = V \\ R_2\dot{q}_2 + \dfrac{q_2}{C_2} = V \\ R_{lea}\dot{q}_{lea} = V \end{cases} \tag{19}$$

The following constraint represents the kirchhoff Current Law (KCL) considering a current generator supplying the supercapacitor:

$$\dot{q}_1 + \dot{q}_2 + \dot{q}_{lea} = I \tag{20}$$

Finally, rearranging (19) the final formulation giving the currents as time derivative of charge is obtained.

$$\begin{cases} \dot{q}_1 = \dfrac{C_0}{2k_cR_1} - \dfrac{\sqrt{C_0^2+4k_cq_1}}{2k_cR_1} + \dfrac{V}{R_1} \\ \dot{q}_2 = -\dfrac{q_2}{R_2C_2} + \dfrac{V}{R_2} \\ \dot{q}_{lea} = \dfrac{V}{R_{lea}} \end{cases} \tag{21}$$

5. Simulink implementation

The model of the supercapacitor based on eqs. (21) has been implemented in Simulink® environment. Figure 2 shows the implementation of (20) and of the third eqs. (21) in which the leackage current is obtained.

The "step" block represents the input current of the supercapacitor. It will be used for a test in which the supercapacitor is charged at constant current and then discharged on the leakage resistance. The voltage at the terminals of the supercapacitor E is obtained as the leakage current multiplied for the leakage resistance R_{lea}.

The implementation of the first equation (21) is shown in figure 3. On the basis of the voltage E obtained by (21) the time derivative of the charge stored in C_1 equivalent capacitor dq_1 is obtained. The implementation of the first equation (21) is shown in figure 4. It uses the voltage E as input and gives the time derivative of the charge stored in C_2 as dq_2.

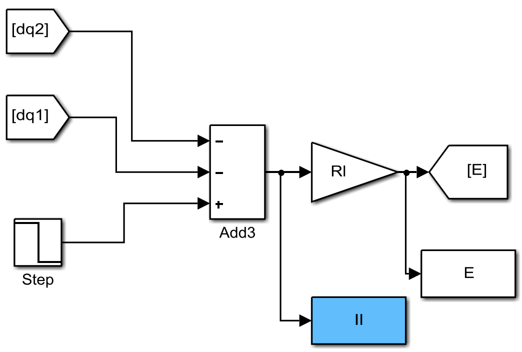

Fig. 2: implementation of the equation (20).

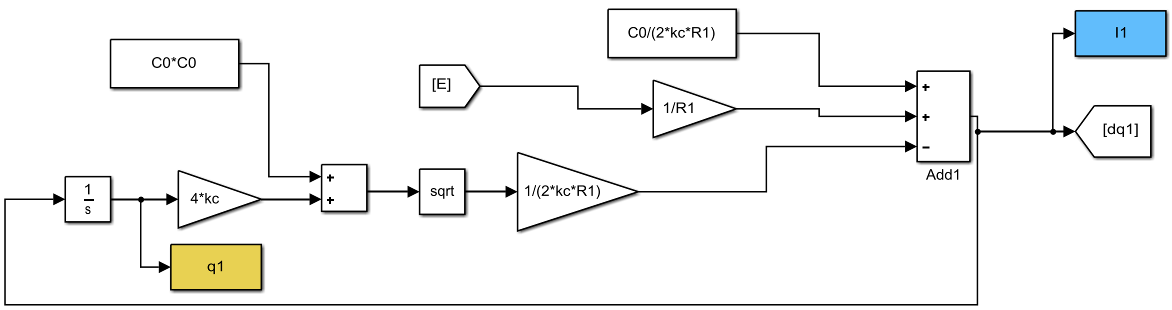

Fig. 3: implementation of the first equation (21).

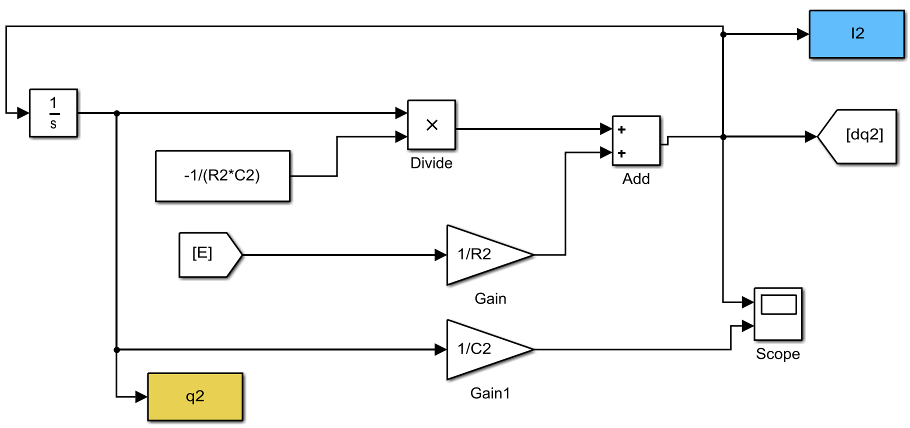

Fig. 4: implementation of the second equation (21).

6. Experimental set-up

A test bench has been arranged to acquire electrical data coming from the supercapacitor under test. A SC-Toolbox for Matlab+Simulink environment, able to analyse the collected data, has been developed. The laboratory setup is composed of a power supply, the supercapacitor, the resistor load the switch and the sensors, a multimeter, the voltage and current sensor board and a laptop. The Main components of the experimental rig are summarized in table I. The supercapacitor is the Maxwell model BMOD0083 P048 B01, it is shown in figure 5, the main characteristic parameters of the supercapacitor under test are the following:

•Rated Capacitance: 83 F
•Minimum Capacitance, initial: 83 F
•Maximum ESRDC, initial: 10 mΩ
•Rated Voltage: 48 V
•Absolute Maximum Voltage: 51 V
•Maximum Continuous Current (ΔT = 15C°): 61 ARMS
•Maximum Continuous Current (ΔT = 40C°): 100 ARMS
•Maximum Peak Current, 1 s. (non-repetitive): 1100 A

Table I: Main components of the experimental rig

supplier	function	code
TDK-Lambda	Power Supply	TDKGEN60-40
Maxwell Technologies	Supercapacitor	BMOD0083P048B01
	Sensor Board	
Sorensen	Electronic Load	SLH500V-6A
RS-Components	Resistor passive Load	RS136-238200W3R3J 01.03
ABB	Double Pole Switch	S8025UCK32500Vdc50kA
Asus	Laptop PC	
MathWorks	Software	MATLAB7.4, Simulink 6.6.1
dSPACE	Hardware Device	dSPACESYSTEM 3.3.0.7
dSPACE	Hardware DAQ Board	RTI2004LIB DS2004ADC
Tektronix	Digital Multimeter	DMM4050 $6^{1}/2$-Digit

The parameters of the SC module have been identified on the basis of the method described in [8-9]

$$R_1 = 0.01\ \Omega, \quad R_2 = 10\ \Omega, \quad R_3 = 1120\ \Omega,$$
$$C_{1a} = 38\ \text{F}, \quad C_{2a} = 13\ \text{F}, \quad K_{1b} = 0.93\ \text{F/V}.$$

Fig. 5: maxwell supercapacitor (left) with the electronic load and the power supply (right)

7. Results

A test with constant current charge and free discharge has been performed both in simulation and experimentally.
The supercapacitor starts from zero voltage condition, it is charged with a constant current until a voltage of about 46 V is reached, then, maintaining opened its terminals, it is discharged on the leakage resistance. For given current value, the charging is stopped at the same instant in simulation and experimental test. The current ranges from 5A to 40 A. AT t=1200s a forced discharge is performed on a 1.1 Ω resistance.
The supply current is shown in figure 6. The measured voltages at the SC terminals are shown in figures 7a and 7b. Figure 7a includes the time interval ranging from 0 to 1200s whereas figure 7b includes the time interval ranging from 1000 to 2000s. The calculated voltages at the SC terminals are shown in figures 8a and 8b. Figure 8a includes the time interval ranging from 0 to 1200s whereas figure 8b includes the time interval ranging from 1000 to 2000s. Comparing these curves, the parameters of the SC have been more precisely tuned to minimize the error between the experimental results and the results obtained by simulation. The relative error is shown in figure 9a and 9b. In figure 9a it can be noted that the relative error rises up to 100% near the time interval corresponding to values of the voltage almost null since it is influenced by these small values. Figure 9b shows the error up to 1200s, it is always lower than 4%.

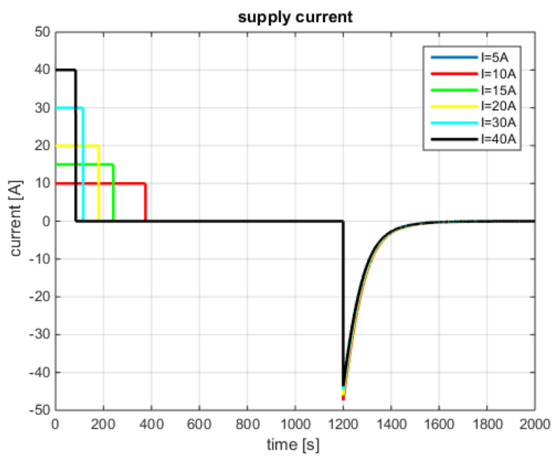

Fig. 6: supply current

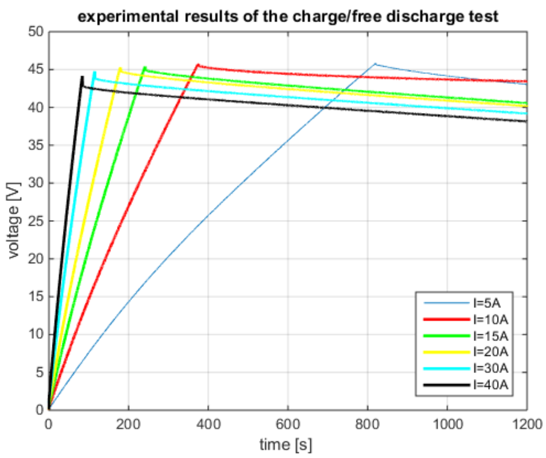

Fig. 7a: measured voltage at the SC terminals

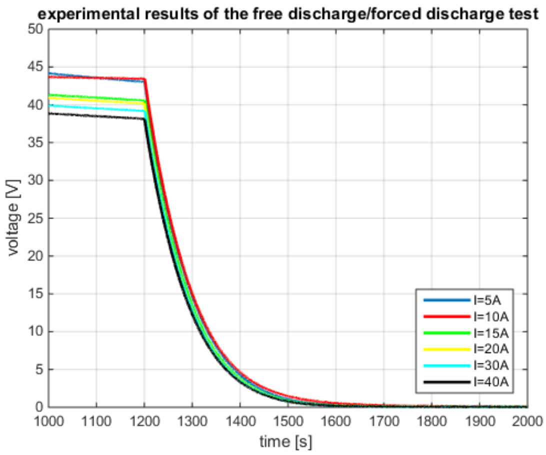

Fig. 7b: measured voltage at the SC terminals

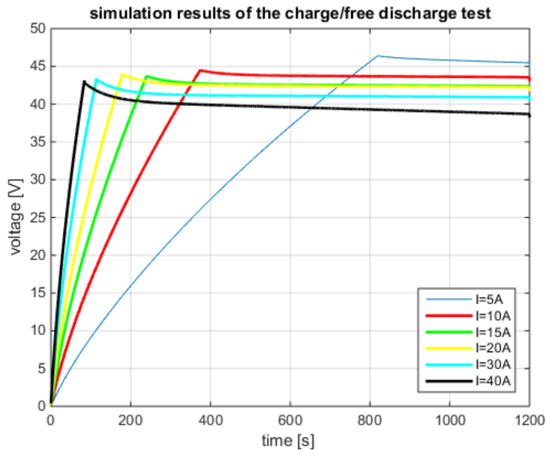

Fig. 8a: calculated voltage at the SC terminals

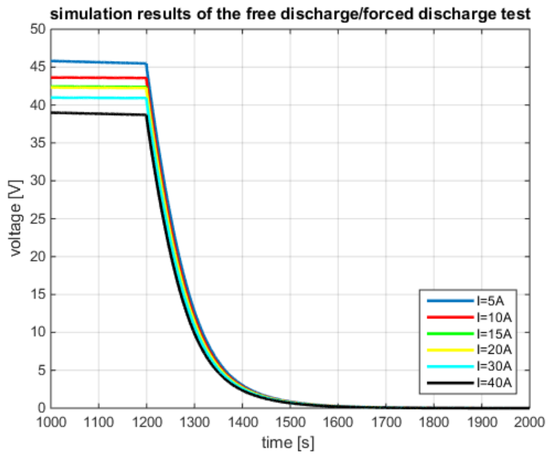

Fig. 8b: calculated voltage at the SC terminals

Once the model has been validated, it has been used to calculate the voltages and the current flowing through the equivalent capacitors C_1 and C_2. The values corresponding to the supply current of 40A (see figure 6) have been calculated. Figure 10 shows the current flowing through the equivalent capacitors C_1 as defined by (1), it can be noted that this is higher than the current of the linear capacitor C_2. Figure 11 shows the current flowing through the equivalent capacitors C_2 it is smoothed compared with the current flowing through C1.

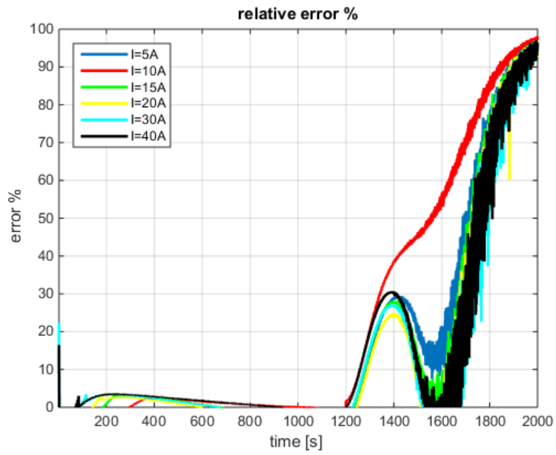

Fig. 9a: relative error between measures and calculated voltage

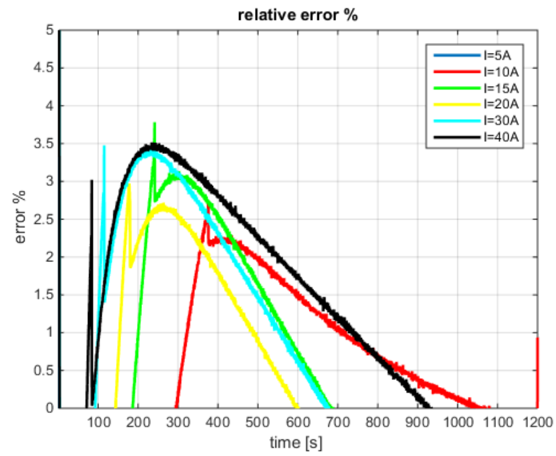

Fig. 9b: relative error between measures and calculated voltage

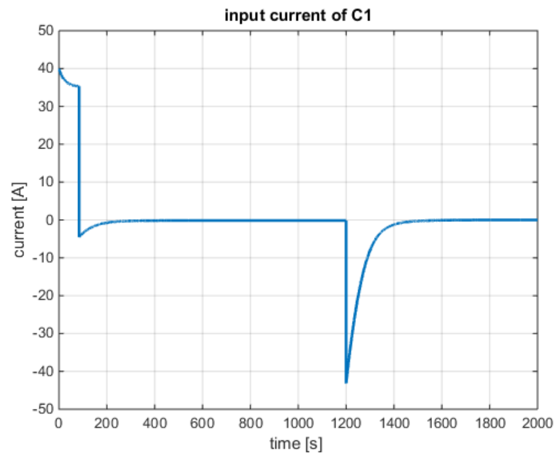

Fig. 10: current flowing through C1

The current flowing through the leakage resistance R_{lea}, shown in figure 12, has the same shape of the voltage at the SC terminals. As far as the voltages are concerned, the voltage at the C_1 terminals follows the voltage at the SC terminals since the value of the parasitic resistance R_1 is very low. The voltage at the C_2 terminals exhibits the same values but is smoothed.

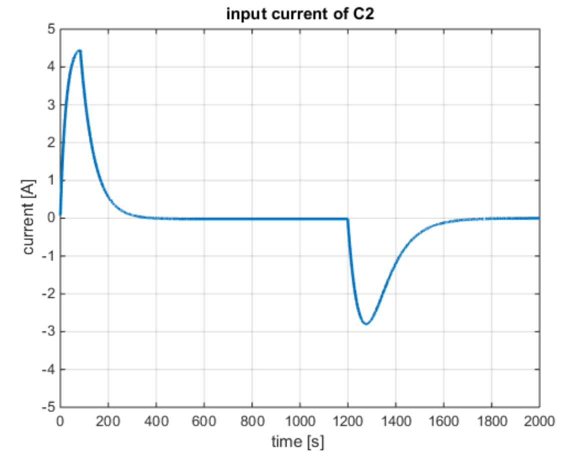

Fig. 11: current flowing through C2

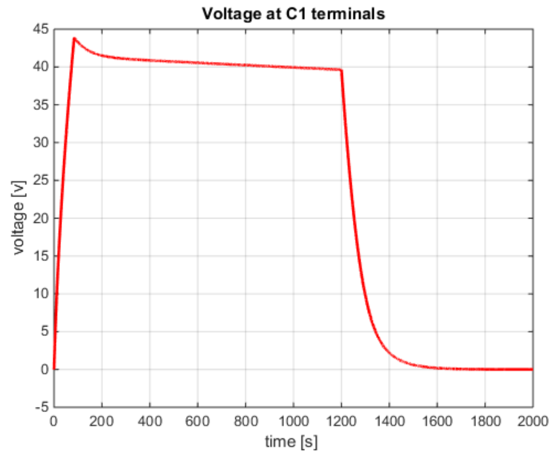

Fig. 13: voltage at the C1 terminals

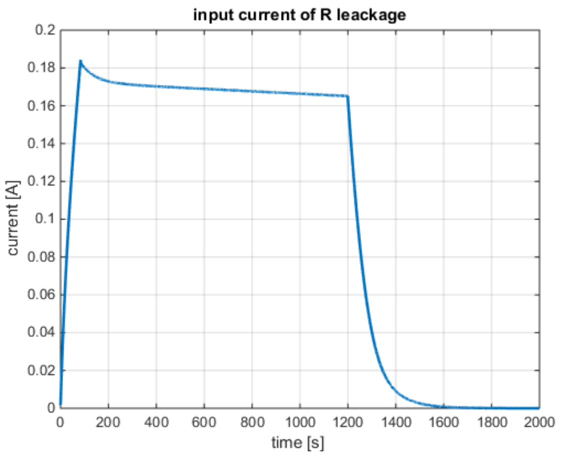

Fig. 12: current flowing through R leakage

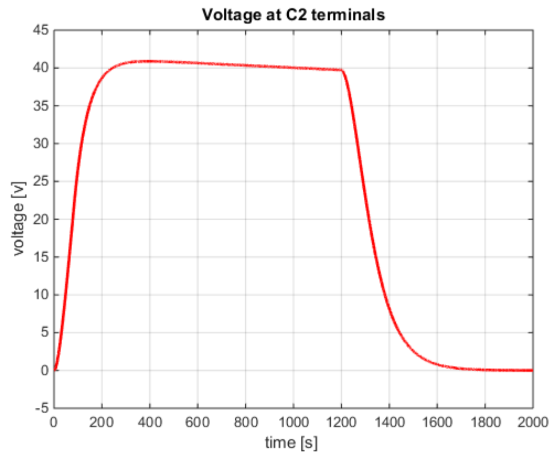

Fig. 14: voltage at the C2 terminals

8. Conclusion

A two branch model of a supercapacitor based on the Lagrange's equation has been developed. This approach uses the energy formulation to obtain the differential equations using the charge formulation. The model has been then implemented in Simulink® environment. The use of Lagrange's equation allows additional terms, as for example the energy coming from a mechanical system, to be easily accounted for in the Lagrange function. This implementation does not require a circuit simulator obtaining a reduced computation time and it is able to reproduce the current and voltages at the equivalent capacitor terminals.

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